Journal of Statistical Physics, Vol. 74, Nos. 5/6, 1994

More Results on the Ashkin–Teller Model

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Received May 22, 1993; final November 5, 1993

We analyze the low-temperature phase diagram of the Ashkin–Teller model for real values of the quadratic and quartic coupling constants.

KEY WORDS: Restricted ensembles; residual entropy; Pirogov–Sinai theory.

1. INTRODUCTION AND RESULTS

The Ashkin–Teller (AT) model is described by two coupled Ising models such that in each site x of the hypercubic lattice $\mathbf{Z}^d$, $d \geqslant 2$, there are two independent spin variables $\sigma_x = \pm 1$ and $\tau_x = \pm 1$. The Hamiltonian in a finite box $\Lambda \subset \mathbf{Z}^d$ is formally defined by

$$H = -\frac{\lambda_1}{2} \sum_{\langle xy \rangle \in \mathbf{L}} \sigma_x \sigma_y - \frac{\lambda_1}{2} \sum_{\langle xy \rangle \in \mathbf{L}} \tau_x \tau_y - \lambda_2 \sum_{\langle xy \rangle \in \mathbf{L}} \sigma_x \sigma_y \tau_x \tau_y \tag{1.1}$$

Here $\langle xy \rangle$ denotes a pair of nearest-neighbor sites and $\mathbf{L}$ is the set of bonds $b = \langle xy \rangle$ in Λ. λ_1 and λ_2 are real coupling constants.

The AT model has been extensively studied by means of approximate and numerical methods (see refs. 1 and 2 and references therein). The ferromagnetic case corresponding to the positive values of λ_1 and λ_2 was solved exactly by Baxter[(3)] in dimension $d = 2$. He obtained that there exist two phase transitions with a symmetry breakdown for some values of λ_1 and λ_2. Baxter's result has been extended to all dimensions $d \geqslant 2$ by Pfister[(4)] by using the correlation inequalities.

Our goal is to analyze the phase diagram and discuss the nature of the pure phases occurring in the AT model at low temperature for certain values of the couplings λ_1 and λ_2 not necessarily positive. The method we

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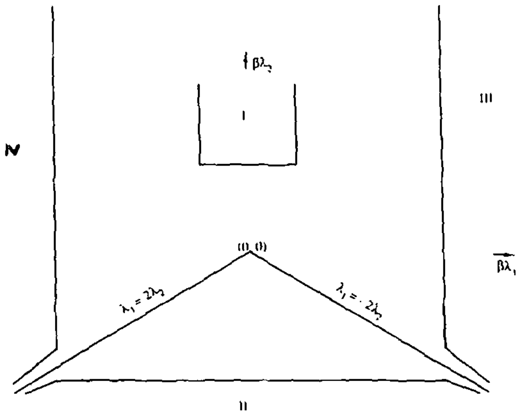

Fig. 1. The low-temperature phase diagram in the plane $(\beta\lambda_1, \beta\lambda_2)$. (a) In region I there are two coexisting phases, (b) in region II two phases coexist, (c) in region III there exist four coexisting ferromagnetic phases, (d) in region IV four antiferromagnetic phases coexist.

use is based on the Pirogov–Sinai (PS) theory[5] and its extension by Bricmont *et al.*[6] (BKL).

To formulate the results we obtain in the plane $(\beta\lambda_1, \beta\lambda_2)$ (see Fig. 1), we first introduce a positive constant $c > 0$ and a large positive parameter μ depending on the dimension d, $\mu = \mu(d)$.

Result 1. For $\beta\lambda_2 > \mu(d)$ and $2\beta\,|\lambda_1| < \ln[d/(d-1)]$: There exist two limiting extremal Gibbs states.

Result 2. For $\beta\lambda_2 < -\mu(d)$ and $|\lambda_1| \leqslant -c - 2\lambda_2$: There exist two limiting extremal Gibbs states.

Result 3. For $\beta\lambda_1 > \mu(d)$ and $\lambda_1 \geqslant c - 2\lambda_2$: There exist four limiting extremal Gibbs states.

Result 4. For $\beta\lambda_1 < -\mu(d)$ and $\lambda_1 \leqslant -c + 2\lambda_2$: There exist four limiting extremal Gibbs states.

2. PROOF OF THE RESULTS

We only sketch the proof of the results announced in Section 1. Concerning result 1, we refer the reader to the BKL theory for details.

First we transform the Ashkin–Teller model described by the Hamiltonian (1.1) into the $Z(4)$ model with the Hamiltonian

$$H = -\lambda_1 \sum_{\langle xy \rangle \in \mathbf{L}} \cos\left(\frac{2\pi(n_x - n_y)}{4}\right) - \lambda_2 \sum_{\langle xy \rangle \in \mathbf{L}} \cos\left(\frac{4\pi(n_x - n_y)}{4}\right) \tag{2.1}$$

Here n_x belongs to the Abelian group $G = \{0, 1, 2, 3\}$ with addition mod(4) as a group law. To obtain the expression (2.1) from the Hamiltonian (1.1) we define a two-dimensional vector

$$S_x = \left(\cos\left(\frac{2\pi n_x}{4}\right), \sin\left(\frac{2\pi n_x}{4}\right)\right)$$

such that

$$\sigma_x = \cos\left(\frac{2\pi n_x}{4}\right) - \sin\left(\frac{2\pi n_x}{4}\right)$$

$$\tau_x = \cos\left(\frac{2\pi n_x}{4}\right) + \sin\left(\frac{2\pi n_x}{4}\right)$$

We will denote by Ω the configuration space, $\Omega = \prod_{x \in \mathbf{Z}^d} G_x$, $G_x = G$; the restriction of Ω to finite box Λ is denoted Ω_Λ.

Finally, we define the Hamiltonian per bond

$$H_b = -\lambda_1 \cos\left(\frac{2(n_x - n_y)\,\pi}{4}\right) - \lambda_2 \cos\left(\frac{4(n_x - n_y)\,\pi}{4}\right) \tag{2.2}$$

2.1. Proof of Result 1

2.1.1. Restricted Ensembles and Diluteness. Suppose that $\lambda_1 = 0$ in (2.2) and $\lambda_2 > 0$; then the configurations $n \in \Omega$ which minimize the energy are the following:

$$n_x - n_y = 0, \qquad \forall \langle xy \rangle \in \mathbf{L} \tag{2.3a}$$

or

$$n_x - n_y = 2, \qquad \forall \langle xy \rangle \in \mathbf{L} \tag{2.3b}$$

We see that there exist two ensembles of configurations which realize the condition (2.3),

$$\Omega_f^1 = \{n \in \Omega \mid n(x) = 0, 2, \forall x \in \mathbf{Z}^d\}$$

$$\Omega_f^2 = \{n \in \Omega \mid n(x) = 1, 3, \forall x \in \mathbf{Z}^d\}$$

Referring to the BKL theory, we obtain two *restricted ensembles* Ω_f^1 and Ω_f^2.

Now we suppose that $\lambda_1 \neq 0$ and we seek the condition to impose on $\beta\,|\lambda_1|$ in order to preserve the restricted ensembles Ω_f^1 and Ω_f^2.

We define the restricted partition function

$$Z_R(\Lambda, \beta \mid \alpha) = \sum_{n \in \Omega_f^\alpha} \exp - \beta H_\Lambda(N) \tag{2.4}$$

By using the identity

$$\cos\left(\frac{2(n_x - n_y)\,\pi}{4}\right) = -\frac{1}{2} + 2\delta(n_x - n_y) - \frac{1}{2}\cos\left(\frac{4(n_x - n_y)\,\pi}{4}\right)$$

where δ is the Kronecker symbol satisfying $\delta(\alpha) = 1$ if $\alpha = 0 \bmod(4)$ and $\delta(\alpha) = 0$ otherwise, we write the Hamiltonian (2.2) as

$$H_b(n) = \lambda_1/2 - (\lambda_2 - \lambda_1/2)\cos[(n_x - n_y)\,\pi] - 2\lambda_1\,\delta(n_x - n_y)$$

The restricted partition function is equal to

$$Z_R(\Lambda, \beta \mid \alpha) = \exp[\beta(\lambda_2 - \lambda_1)\,L(\Lambda)] \sum_{n \in \Omega_f^\alpha} \exp\left[2\beta\lambda_1 \sum_{\langle xy \rangle \in \mathbf{L}} \delta(n_x - n_y)\right] \tag{2.5}$$

where $L(\Lambda)$ is the number of bounds in Λ.

Now we consider the last expression in (2.5),

$$\sum_{n \in \Omega_f^\alpha} \exp\left[2\beta\lambda_1 \sum_{\langle xy \rangle \in \mathbf{L}} \delta(n_x - n_y)\right]$$

Suppose that $\beta\lambda_1$ $(\lambda_1 > 0)$ is large; it follows that the configurations which minimize the energy are such that $n_x - n_y = 0$ for every bond $\langle xy \rangle$ and then the condition (2.3b) is not satisfied.

Let $\beta\,|\lambda_1|$ $(\lambda_1 < 0)$ be large; then the configurations satisfying $n_x - n_y = 2$ for every bond $\langle xy \rangle$ minimize the energy and the condition (2.3a) fails

We conclude that for $\beta\,|\lambda_1|$ large the conditions (2.3a) and (2.3b) are not compatible.

To preserve the condition (2.3), i.e., to get a control of the excitations (in terms of $\beta\lambda_1$) which "destabilize" the restricted ensembles Ω_f^1 and Ω_f^2 corresponds to verifying the *diluteness property* introduced in BKL.

The diluteness property means that we need a convergent cluster expansion (strong clustering properties) for the restricted ensembles in order to prove the existence of their free energies $f_R(\alpha)$,

$$f_R(\alpha) = -\frac{1}{\beta} \lim_{\mathbf{L} \to \infty} \frac{1}{L(\Lambda)} \ln Z_R(\Lambda, \beta \mid \alpha) \tag{2.6}$$

To prove the diluteness property we use the "high-temperature expansion" for the partition function

$$Z_R(\Lambda, \beta \mid \alpha) = \sum_{n \in \Omega_f^\alpha} \exp\left[2\beta\lambda_1 \sum_{\langle xy \rangle \in \mathbf{L}} \delta(n_x - n_y)\right] \tag{2.7}$$

Since for every configuration $n \in \Omega_f^\alpha$, the variable n_x takes only two values, then (2.7) is the partition function of the Ising model obtained by using the identity

$$\delta(n_x - n_y) = \frac{1 + \sigma_x \sigma_y}{2}, \qquad \sigma_x = \pm 1$$

Therefore (2.7) is reduced to

$$\exp[\beta\lambda_2 L(\Lambda)] \sum_{\{\sigma\}} \exp\left(\beta\lambda_1 \sum_{\langle xy \rangle \in \mathbf{L}} \sigma_x \sigma_y\right) \tag{2.8}$$

and it is known that for $|\tanh \beta\lambda_1|\,(2d-1) < 1$ the expression (2.8) satisfies a convergent cluster expansion.

It follows that once the condition $|\beta\lambda_1| < \ln\{[d/(d-1)]^{1/2}\}$ is satisfied, we use the algebraic formalism[8,9] to prove that the free energy defined in (2.6) exists and the restricted ensembles Ω_f^α, $\alpha = 1, 2$, are dilute.

2.1.2. The Low-Temperature Expansion. To prove that there exist two phases corresponding to the low-temperature excitations of the two restricted ensembles Ω_f^α, $\alpha = 1, 2$ we use the PS Theory and its BKL extension. In this formalism the low-temperature excitations are generated by contours separating regions in Λ with boundary conditions in Ω_f^α, $\alpha = 1, 2$. These excitations will be controled whenever the Peierls condition (an exponential decay of the probability of a contour with its length) is verified.

To define contours we introduce the following notations. For a given configuration $n \in \Omega$, a bond $\langle xy \rangle$ is *regular* if the restriction of this configuration to the sites x and y belongs to the same restricted ensemble Ω_f^α, $\alpha = 1, 2$. Otherwise the bond $\langle xy \rangle$ is *irregular*. The set $B(n)$ of irregular bonds in n forms a geometric configuration of bonds in $\mathbf{L}$. Its dual $B^*(n)$ is decomposed into its connected components, which we call contours.

Taking into account that $\delta(n_x - n_y) = 0$ and $\cos \pi(n_x - n_y) = -1$ for all bonds in $B(n)$ and considering the symmetry $Z(\Lambda, \beta \mid 1) = Z(\Lambda, \beta \mid 2)$, one easily reduces the system to a contour model with weak interactions controlled by the convergent cluster expansion in (2.8), and the Peierls inequality for the contour correlation function is obtained by the same method in the interacting contour model (Appendix 2 in BKL).

One proves that there exist two limiting extremal Gibbs states P_1 and P_2 such that

$$P_1(n_x = 0 \text{ or } 2) > \tfrac{1}{2} \qquad \text{and} \qquad P_2(n_x = 1 \text{ or } 3) > \tfrac{1}{2}$$

2.2. Proof of Result 2

2.2.1. Restricted Ensembles. Let $2\lambda_2 < -|\lambda_1|$; then there exist two restricted ensembles Ω^1_{af} and Ω^2_{af} obtained as follows: We consider a bipartite lattice Λ defined as the union of two sublattices Λ_1 and Λ_2 such that for each site $x \in \Lambda_1$ (resp. $x \in \Lambda_2$) the neighbor sites V_x of x belong to Λ_2 (resp. Λ_1). We also introduce the set $X = \{0, 2\}$ and $Y = \{1, 3\}$, to define

$$\Omega^\alpha_{af} = \{n \in \Omega \mid \forall \langle xy \rangle, n_x \in X, x \in \Lambda_\alpha \text{ and } n_y \in Y, y \in \Lambda_\delta, \alpha \neq \delta\}$$

Since

$$\cos\left(\frac{(n_x - n_y)\pi}{2}\right) = 0, \qquad \forall n \in \Omega^\alpha_{af}$$

therefore the restricted partition function with the Hamiltonian (2.1) is

$$\begin{aligned} Z_R(\Lambda, \beta \mid \alpha) &= \sum_{n \in \Omega^\alpha_{af}} \exp - \beta H_\Lambda(n) \\ &= \exp[-\beta\lambda_2 L(\Lambda)] \sum_{n \in \Omega^\alpha_{af}} 1 \\ &= 2^{|\Lambda|} \exp[-\beta\lambda_2 L(\Lambda)] \end{aligned} \tag{2.9}$$

Hence Ω^α_{af} is the class of the ground states with the *residual entropy* (the entropy at zero temperature) considered in ref. 7. We notice that it follows from the formula (2.9) that the residual entropy per site is ln 2.

2.2.2. The Low-Temperature Expansion. In this section we introduce the notion of *regular* and *irregular* plaquettes.

For a given configuration $n \in \Omega$, a plaquette p is regular if the restriction of the configuration n to the sites which are endpoints of bonds in the boundary of the plaquette p belongs to the restricted ensemble Ω^α_{af}. Otherwise the plaquette p is irregular.

We denote by $\mathscr{P}(n)$ the set of irregular plaquettes in n and by $\mathscr{L}(n)$ the set of bonds in the boundaries of the plaquettes in $\mathscr{P}$. Finally we define $\mathscr{S}(n)$ as the set of sites which are endpoints of the bonds in $\mathscr{L}(n)$.

The set $\mathscr{B}(n) = (\mathscr{P}(n), \mathscr{L}(n), \mathscr{S}(n))$ is decomposed into its connected components, which we call contours. Here a contour γ is a pair, $\gamma = (\Gamma, n_\Gamma)$, of its support Γ and the configuration n_Γ restricted to this support.

Since the configurations are specified on the sites in the boundary of Γ and belong to the restricted ensemble Ω^{α}_{af}, then

$$\cos\left(\frac{(n_x - n_y)\,\pi}{2}\right) = 0, \qquad x \in \mathscr{S}(\Gamma), \quad y \in \Lambda \backslash \mathscr{S}(\Gamma)$$

Here $\mathscr{S}(\Gamma)$ is the set of sites in Γ. Moreover, we obtain a noninteracting contour model as in the standard PS theory.

To verify the Peierls condition it suffices to get an upper bound on the expression

$$2^{-|\Gamma|} \exp[\beta\lambda_2 L(\Gamma)]\, Z(\Gamma, \beta \mid \nu) \tag{2.10}$$

Here the partition function is

$$Z(\Gamma, \beta \mid \nu) = \sum_{n \in \Omega_l} \nu_\Gamma(n) \exp - \beta H_\Gamma(n) \tag{2.11}$$

and $\nu_\Gamma(n)$ is the characteristic function that Γ is the support of a contour γ in the configuration n.

To obtain an upper bound for the expression (2.11) we introduce the following definitions.

A bond $\langle xy \rangle$ is a *regular* bond if it satisfies

$$n_x \in X \text{ (resp. } n_x \in Y) \qquad \text{and} \qquad n_y \in Y \text{ (resp. } n_y \in X)$$

A bond $\langle xy \rangle$ is an *excited* bond if

$$n_x \in X \text{ (resp. } n_x \in Y) \qquad \text{and} \qquad n_y \in X \text{ (resp. } n_y \in Y)$$

We denote by $E(\Gamma)$ (resp. $W(\Gamma)$) the number of excited (resp. regular) bonds in Γ. In fact, $L(\Gamma) = E(\Gamma) + W(\Gamma)$.

Since for an excited bond

$$\cos\left(\frac{(n_x - n_y)\,\pi}{2}\right) = \pm 1$$

we get

$$Z(\Gamma, \beta \mid \nu) \leqslant 2^{|\Gamma|} \exp\{-\beta\lambda_2[W(\Gamma) - E(\Gamma)]\} \exp\{\beta\, |\lambda_1|\, E(\Gamma)\}$$

Therefore the expression (2.10) is bounded by

$$\exp[\beta(|\lambda_1| + 2\lambda_2)\, E(\Gamma)] \tag{2.12}$$

Now we introduce a positive constant $c > 0$ such that

$$-|\lambda_1| \geqslant 2\lambda_2 + c \tag{2.13}$$

to obtain that the expression (2.12) is bounded by

$$\exp\, -\beta c E(\Gamma) \tag{2.14}$$

Since each plaquette p of Γ contains 2 or 4 excited bonds of Γ, we get a bound in terms of the number of plaquettes and the number of bonds in Γ,

$$E(\Gamma) \geqslant \frac{P(\Gamma)}{d-1} \geqslant \frac{L(\Gamma)}{2(d-1)}$$

Therefore (2.10) is bounded by

$$\exp\, -\beta c \frac{L(\Gamma)}{2(d-1)} \tag{2.15}$$

and the Peierls condition is verified whenever β is large.

We conclude that whenever the conditions (2.13) and (2.15) are satisfied there exist two phases corresponding to small perturbations of the restricted ensembles Ω^{α}_{af}, $\alpha = 1, 2$

Proof of Results 3 and 4

1. For

$$\lambda_1 > 0 \qquad \text{and} \qquad \lambda_1 + 2\lambda_2 > 0$$

there exist four ferromagnetic ground states, which we denote

$$\Omega^{\alpha}_{f} = \{n \in \Omega \mid n_x = \alpha,\ \forall x \in \mathbf{Z}^d\}$$

2. For

$$\lambda_1 < 0 \qquad \text{and} \qquad -\lambda_1 + 2\lambda_2 > 0$$

there exist four antiferromagnetic ground states defined as follows: Defining the sets $X = \{0, 2\}$ and $Y = \{1, 3\}$ and considering a bipartite lattice $\Lambda = \Lambda_1 \cup \Lambda_2$, we obtain

$$\Omega^{\alpha}_{af}(X) = \{n \in \Omega \mid n_x = 0,\ x \in \Lambda_\alpha \text{ and } n_y = 2,\ y \in \Lambda_\delta,\ \alpha \neq \delta,\ \forall \langle xy \rangle \in \mathbf{L}\}$$

$$\Omega^{\alpha}_{af}(Y) = \{n \in \Omega \mid n_x = 1,\ x \in \Lambda_\alpha \text{ and } n_y = 3,\ y \in \Lambda_\delta,\ \alpha \neq \delta,\ \forall \langle xy \rangle \in \mathbf{L}\}$$

The contours are defined as in Section 2.2.1 and the standard Peierls condition is satisfied for β large and $|\lambda_1| + 2\lambda_2 \geqslant c$, $c > 0$.

3. CONCLUSION AND OPEN PROBLEMS

1. The line $\lambda_1 = 2\lambda_2$ with $\lambda_1 < 0$ and $\lambda_2 < 0$ corresponds to the antiferromagnetic four-state Potts model, for which it was argued that, for high dimensions, the truncated correlation functions go from an exponential decay (in the high-temperature regime) to an algebraic power law decay (in the low-temperature regime). This is based on numerical computations performed by Berker and Kadanoff[10] for the antiferromagnetic q-state Potts models for $q \geqslant 3$; we refer the reader to ref. 11 for a review of recent results.

2. On the line $\lambda_1 = -2\lambda_2$ with $\lambda_1 > 0$ and $\lambda_2 < 0$, it seems that for high dimensions ($d > 2$) there exist two different temperature regimes[1]; nevertheless the result from ref. 2 disagrees with the conclusions of ref. 1. Still considering a bipartite lattice, $\Lambda = \Lambda_1 \cup \Lambda_2$, if we replace n_x by $n_x - 2$ for each site $x \in \Lambda_1$, we obtain the antiferromagnetic four-state Potts model; then the two lines discussed above are related by a symmetry.

3. For very low temperatures, we show that the two lines are *asymptotic* to the boundaries of the phase diagram we obtain (see Fig. 1).

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